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Doubly nonlocal Fisher–KPP equation: Front propagation

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Abstract

We study propagation over $\mathbb{R}^d$ of the solution to a doubly nonlocal reaction-diffusion equation of the Fisher–KPP-type with anisotropic kernels. We present both necessary and sufficient conditions which ensure linear in time propagation of the solution in a direction. For kernels with a finite exponential moment over $\mathbb{R}^d$ we prove front propagation in all directions for a general class of initial conditions decaying in all directions faster than any exponential function (that includes, for the first time in the literature about the considered type of equations, compactly supported initial conditions).

Keywords: nonlocal diffusion, Fisher–KPP equation, nonlocal nonlinearity, long-time behavior, front propagation, anisotropic kernels, integral equation

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1 Introduction

We will study front propagation of solutions to the equation

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \varkappa^+ \int_{\mathbb{R}^d} a^+(x-y)u(y,t)dy - mu(x,t) - u(x,t)\, G\bigl(u(x,t)\bigr),\\ G\bigl(u(x,t)\bigr) &:= \varkappa_\ell u(x,t) + \varkappa_{n\ell} \int_{\mathbb{R}^d} a^-(x-y)u(y,t)dy.\end{aligned} \tag{1.1}$$

Here $d \in \mathbb{N}$; $\varkappa^+, m > 0$ and $\varkappa_\ell, \varkappa_{n\ell} \geq 0$ are constants, such that

$$\varkappa^- := \varkappa_\ell + \varkappa_{n\ell} > 0;$$

the kernels $0 \leq a^\pm \in L^1(\mathbb{R}^d)$ are probability densities, i.e. $\int_{\mathbb{R}^d} a^\pm(y)dy = 1$.

In order to exclude a trivial long-time behavior of the solution to (1.1), we assume

$$\varkappa^+ > m. \tag{A1}$$

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The solution $u = u(x,t)$ describes the local density of a species at the point $x \in \mathbb{R}^d$ at the moment of time $t \geq 0$. The individuals of the species spread over the space $\mathbb{R}^d$ according to the dispersion kernel a^+ and the fecundity rate $\varkappa^+$. The individuals may die according to both constant mortality rate m and density dependent competition, described by the rate $\varkappa^-$. The competition may be *local*, when the density $u(x,t)$ at a point x is influenced by itself only, with the rate $\varkappa_\ell$, or *nonlocal*, when the density $u(x,t)$ is influenced by all values $u(y,t)$, $y \in \mathbb{R}^d$, averaged over $\mathbb{R}^d$ according to the competition kernel a^- with the rate $\varkappa_{n\ell}$. For further motivations and derivation of (1.1) see [3, 7, 9, 10, 16, 22, 23, 26]

One can rewrite then the equation (1.1) in the reaction-diffusion form

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x-y)\big(u(y,t) - u(x,t)\big)dy \\ + u(x,t)\Big(\beta - G\big(u(x,t)\big)\Big),\end{aligned} \tag{1.2}$$

where $\beta = \varkappa^+ - m > 0$.

The first summand here describes a non-local diffusion generator, see e.g. [2] (also known as the generator of a continuous time random walk in $\mathbb{R}^d$ or of a compound Poisson process on $\mathbb{R}^d$). As a result, the solution u to (1.2) may be interpreted as a density of a species which invades according to a nonlocal diffusion within the space $\mathbb{R}^d$ meeting a reaction $Fu := u(\beta - Gu)$; see e.g. [8, 24, 28]. We treat then (1.1) as a doubly nonlocal Fisher–KPP equation, see the introduction to [12] for details.

From (A1) it follows, that the equation (1.1) has two constant stationary solutions: $u \equiv 0$ and $u \equiv \theta$, where

$$\theta := \frac{\varkappa^+ - m}{\varkappa^-} > 0.$$

Under (A1) the following assumption ensures the comparison principle for the equation (1.1), see Proposition 2.2 below:

$$\varkappa^+ a^+(x) \geq \varkappa_{n\ell}\theta a^-(x), \quad \text{a.a. } x \in \mathbb{R}^d. \tag{A2}$$

In particular, (A1)–(A2) imply that the inequality $0 \leq u(x,t) \leq \theta$ a.e. in x holds for all $t > 0$ provided that it holds for $t = 0$. Note that (A2) is evidently fulfilled if, say, $\varkappa_{n\ell} = 0$ or $a^- = a^+$.

Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$ centered at the origin. For an arbitrary direction $\xi \in S^{d-1}$, we define

$$\mathfrak{a}_\xi(\lambda) := \int_{\mathbb{R}^d} a^+(x)e^{\lambda x\cdot\xi}\,dx \in (0,\infty], \quad \lambda > 0. \tag{1.3}$$

Henceforth, by $x \cdot \xi$ we denote the scalar product in $\mathbb{R}^d$. To ensure finite speed of the propagation in a given direction, we assume that, for the fixed $\xi \in S^{d-1}$,

$$\text{there exists } \mu = \mu(\xi) > 0 \text{ such that } \mathfrak{a}_\xi(\mu) < \infty. \tag{A3$_\xi$}$$

For each $\xi \in S^{d-1}$, we denote

$$c_*(\xi) = \begin{cases} \inf\limits_{\lambda>0} \dfrac{\varkappa^+\mathfrak{a}_\xi(\lambda) - m}{\lambda}, & \text{if (A3$_\xi$) holds,} \\ \infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

Consider the set

$$\Upsilon_* := \bigl\{x \in \mathbb{R}^d \mid x \cdot \xi \le c_*(\xi),\ \xi \in S^{d-1}\bigr\}. \tag{1.5}$$

Clearly, Υ_* is a closed convex subset of $\mathbb{R}^d$. In particular, if (A3$_\xi$) fails for all $\xi \in S^{d-1}$, then $\Upsilon_* = \mathbb{R}^d$.

In the sequel, we denote $tA := \{tx \mid x \in A\}$ for a measurable set $A \subset \mathbb{R}^d$, and $\operatorname{int}(A)$ means the interior of a closed set $A \subset \mathbb{R}^d$. Let also $B_\rho(y)$ denote the ball in $\mathbb{R}^d$ of the radius $\rho > 0$ centered at the point $y \in \mathbb{R}^d$.

The following theorem is the main result of the paper; it states, informally, that, for a solution $u(x,t)$ to (1.1) and for any $\varepsilon > 0$, the function $u(tx,t)$ converges (as $t \to \infty$) to θ locally uniformly on the set $(1-\varepsilon)\Upsilon_*$ and converges to 0 locally uniformly out of the set $(1+\varepsilon)\Upsilon_*$. Moreover, if Υ_* is bounded, then both convergences hold uniformly.

Theorem 1.1. *Let the conditions* (A1)–(A2) *hold. Suppose also that*

$$a^+ \in L^\infty(\mathbb{R}^d), \tag{A4}$$

$$\int_{\mathbb{R}^d} |x| a^+(x)\, dx < \infty, \tag{A5}$$

$$\begin{gathered} \text{there exists } \rho, \delta > 0, \text{ such that} \\ \varkappa^+ a^+(x) - \varkappa_{n\ell}\theta a^-(x) \ge \rho \text{ for a.a. } x \in B_\delta(0). \end{gathered} \tag{A6}$$

Let $0 \le u_0(x) \le \theta$ for a.a. $x \in \mathbb{R}^d$, and let $u = u(x,t)$ be the solution to (1.1) *on $\mathbb{R}_+ := [0,\infty)$ such that $u(x,0) = u_0(x)$ (see Subsection 2.1 below for details).*

1. *Let there exist $\xi \in S^{d-1}$ such that* (A3$_\xi$) *holds. Let u_0 be such that, for all $\lambda > 0$,*

$$\operatorname*{esssup}_{x \in \mathbb{R}^d} u_0(x) e^{\lambda x \cdot \xi} < \infty \tag{1.6}$$

for all those $\xi \in S^{d-1}$ where $c_*(\xi) < \infty$. Then, for any compact set $\mathscr{C} \subset \mathbb{R}^d \setminus \Upsilon_*$,

$$\lim_{t \to \infty} \operatorname*{esssup}_{x \in t\mathscr{C}} u(x,t) = 0. \tag{1.7}$$

If, additionally, the set Υ_* is bounded (e.g. if (A7) *below holds), then* (1.7) *holds for any (unbounded) closed set $\mathscr{C} \subset \mathbb{R}^d \setminus \Upsilon_*$.*

2. *Let $u_0(x) \ge \eta$ for a.a. $x \in B_r(x_0)$ with some $x_0 \in \mathbb{R}^d$ and $\eta, r > 0$. Then, for any compact set $\mathscr{C} \subset \operatorname{int}(\Upsilon_*)$,*

$$\lim_{t \to \infty} \operatorname*{essinf}_{x \in t\mathscr{C}} u(x,t) = \theta. \tag{1.8}$$

Remark 1.2. 1. Besides equation (1.1) is a straightforward nonlocal analogue to the classical Fisher–KPP equation (see a discussion in [12, Introduction]), it can't be covered by the existing in the literature methods and results even for more simple case of the local reaction (when $\varkappa_{n\ell} = 0$). For example, it seems to be a nontrivial problem to check whether the flow generated by the equation (1.1) (see Subsection 2.1 below for details) is compact, and hence numerous corresponding techniques, see e.g. [20, 21], can't be applied (cf. also similar discussion in [27, Introduction]). On the other hand, the results obtained in [27] for a non-homogeneous generalization of the local (1.1), having in the background Weinberger's technique [32], had crucial restriction on the initial condition u_0. In particular, a compactly supported u_0 was not allowed at all (even for the considered in [27] a compactly supported kernel a^+). In the present paper the remove this limitation of Weinberger's approach for initial conditions to the so-called monostable reaction-diffusion equations. Note also that compactly supported initial conditions were allowed in [37], however, the technique proposed there worked for the so-called degenerate reaction only, which corresponds to $G(u) = \varkappa_\ell u^p$ with $p > 1$ in (1.1).

2. Informally, front for (1.1) is a set which separates $\mathscr{C} \subset \mathbb{R}^d$, where $u(tx,t) \to \theta$, $x \in \mathscr{C}$, $t \to \infty$, and $\mathscr{O} \subset \mathbb{R}^d$, where $u(tx,t) \to 0$, $x \in \mathscr{O}$, $t \to \infty$, cf. e.g. [25, 34]. The results of Theorem 1.1 show that any ε-neighborhood of the boundary of Υ_* can be considered as a front set.

3. By Proposition 4.2 below, a sufficient condition that Υ_* is a bounded (and hence compact) set is that
$$\text{there exists } \mu_d > 0 \text{ such that } \int_{\mathbb{R}^d} a^+(x) e^{\mu_d |x|}\, dx < \infty. \tag{A7}$$
Evidently, (A7) implies (A5). We will show in Remark 4.4, that (A7) is equivalent to that (A3$_\xi$) holds for all $\xi \in S^{d-1}$ or just for all $\xi \in \{e_i, -e_i \mid 1 \le i \le d\}$ with an arbitrary orthonormal basic $\{e_i \mid 1 \le i \le d\}$ in $\mathbb{R}^d$.

4. Note also that, for the first item of Theorem 1.1, it is enough to assume instead of (A6), that
$$\text{there exists } \rho, \delta > 0 \text{ such that } a^+(x) \ge \rho \text{ for a.a. } x \in B_\delta(0). \tag{A8}$$
Moreover, in Proposition 4.7 below we will enhance (1.7) by proving that there exist $\nu = \nu(\mathscr{C}) > 0$ and $D = D(u_0, \mathscr{C}) > 0$, such that
$$\operatorname*{esssup}_{x \in t\mathscr{C}} u(x,t) \le D e^{-\nu t}, \quad t > 0. \tag{1.9}$$

5. By [15], if u_0 is continuous on $\mathbb{R}^d$, then $u(\cdot,t)$ is also continuous on $\mathbb{R}^d$ for all $t > 0$, and one can replace esssup/essinf in (1.7), (1.8) by max/min, correspondingly (note that in the second item of Theorem 1.1 we shall assume then that $u_0 \not\equiv 0$).

6. Note that the assumption (A2) is redundant for the case of the local nonlinear part in (1.1), i.e. where $\varkappa_{n\ell} = 0$. In contrast, if $\varkappa_{n\ell} > 0$ and (A2) fails, the bifurcation of the constant solution $u \equiv \theta$ is possible, developing an infinite family of spatially periodic stationary solutions (see [18] for more details).

7. Recall that if (A3$_\xi$) fails for all $\xi \in S^{d-1}$, i.e. if
$$\int_{\mathbb{R}^d} a^+(x) e^{\lambda x \cdot \xi}\, dx = \infty, \quad \lambda > 0,\ \xi \in S^{d-1}. \tag{1.10}$$
then $\Upsilon_* = \mathbb{R}^d$, i.e. the convergence (1.8) holds for any compact $\mathscr{C} \subset \mathbb{R}^d$.

8. Stress that we actually allow that the initial condition u_0 has exponential decaying, i.e. that (1.6) holds for *some* appropriate $\lambda = \lambda(\xi) > 0$ only, see below Remark 1.4 and Subsection 4.1 for details. Up to our knowledge, this kind of results is new for reaction-diffusion equations.

Theorem 1.1 describes the propagation to all directions. It is based on the properties of the propagation to each direction, that naturally require less restrictive assumptions. Namely, one can weaken the assumptions (A5) and (A8) by assuming that, for a fixed $\xi \in S^{d-1}$,

$$\int_{\mathbb{R}^d} |x \cdot \xi| \, a^+(x) \, dx < \infty, \tag{A9$_\xi$}$$

and

$$\begin{gathered} \text{there exist } r, \rho, \delta > 0 \text{ (depending on } \xi) \text{ such that} \\ a^+(x) \geq \rho \text{ for a.a. } x \in B_\delta(r\xi). \end{gathered} \tag{A10$_\xi$}$$

We set

$$\Upsilon_*(\xi) := \bigl\{ x \in \mathbb{R}^d \mid x \cdot \xi \leq c_*(\xi) \bigr\}, \quad \xi \in S^{d-1}, \tag{1.11}$$

then, in particular, cf. (1.5),

$$\Upsilon_* = \bigcap_{\xi \in S^{d-1}} \Upsilon_*(\xi).$$

Under assumption (A3$_\xi$), we define, see [13] for details,

$$\sigma_\xi(a^+) := \sup\bigl\{ \lambda > 0 \bigm| \mathfrak{a}_\xi(\lambda) < \infty \bigr\} \in (0, \infty].$$

Under the assumption (A9$_\xi$), we define also

$$\mathfrak{m}_\xi := \varkappa^+ \int_{\mathbb{R}^d} x \cdot \xi \; a^+(x) \, dx. \tag{1.12}$$

Theorem 1.3. *Let* (A1), (A2), (A4) *hold. Let* $\xi \in S^{d-1}$ *be fixed, and suppose that* (A3$_\xi$), (A9$_\xi$), (A10$_\xi$) *hold. Then the following statements hold.*

1. *(Cf. [13, Theorem 1.2]) There exists a unique*

$$\lambda_* = \lambda_*(\xi) \in (0, \infty), \qquad \lambda_*(\xi) \leq \sigma_\xi(a^+),$$

such that

$$c_*(\xi) = \min_{\lambda > 0} \frac{\varkappa^+ \mathfrak{a}_\xi(\lambda) - m}{\lambda} = \frac{\varkappa^+ \mathfrak{a}_\xi(\lambda_*) - m}{\lambda_*} > \mathfrak{m}_\xi. \tag{1.13}$$

2. *Let* $0 \leq u_0 \leq \theta$ *be such that* (1.6) *holds true for all* $0 < \lambda < \lambda_*(\xi)$. *Let* u *be the corresponding solution to* (1.1) *on* $\mathbb{R}_+$. *Let* $\mathscr{O}_\xi \subset \mathbb{R}^d$ *be an open set, such that* $\Upsilon_*(\xi) \subset \mathscr{O}_\xi$ *and* $\delta := \operatorname{dist}(\Upsilon_*(\xi), \mathbb{R}^d \setminus \mathscr{O}_\xi) > 0$. *Then the following estimate holds*

$$\operatorname*{esssup}_{x \notin t\mathscr{O}_\xi} u(x,t) \leq \|u_0\|_{\lambda_*, \xi} e^{-\lambda_* \delta t}, \quad t > 0. \tag{1.14}$$

Remark 1.4. 1. In other words, the function $u(tx,t)$ converges (when $t\to\infty$) to 0 uniformly on the hyperspace $\{x\cdot\xi \geq (1+\varepsilon)c_*(\xi)\}$ for each $\varepsilon>0$.

2. We will show in Subsection 4.1, that, similarly to the assumptions of Theorem 1.3, the first item of Theorem 1.1 remains true if (1.6) holds for $0<\lambda\leq\lambda_*(\xi)$ only (for all those ξ where (A3$_\xi$) holds).

Corollary 1.5 (cf. also [4, 14, 17])**.** *Let* (A1) *and* (A2) *hold. Let* $0\leq u_0\leq\theta$ *be such that* (1.6) *holds for all* $\lambda>0$*. Then the assumption* (A3$_\xi$) *is necessary and sufficient to obtain a constant speed of propagation for the corresponding solution to* (1.1) *in the direction* $\xi\in S^{d-1}$.

Under (A5), we define, cf. (1.12),

$$\mathfrak{m} := \varkappa^+ \int_{\mathbb{R}^d} x a^+(x)\,dx \in \mathbb{R}^d. \tag{1.15}$$

By Proposition 4.1 below, if (A1), (A2), (A4)–(A6) hold and if, for some $\xi\in S^{d-1}$, (A3$_\xi$) holds, then

$$\mathfrak{m}\in \operatorname{int}(\Upsilon_*).$$

On Figure 1, we sketched a relation between the sets $\Upsilon_*(\xi)$ and Υ_*. The arrows describe the 'motion' of the sets $t\Upsilon_*(\xi)$ and $t\Upsilon_*$, correspondingly. Note that the origin may be out of $\Upsilon_{*,\xi}$, for some $\xi\in S^{d-1}$, and hence out of Υ_*. However, by Remark 4.5 below, for each $\xi\in S^{d-1}$, the origin must belong to at least one of the sets $\Upsilon_*(\xi)$ and $\Upsilon_*(-\xi)$. A sufficient condition for the inclusion $0\in\operatorname{int}\Upsilon_*$ is e.g. $a^+(-x)=a^+(x)$, $x\in\mathbb{R}^d$ (then $\mathfrak{m}=0$).

Figures 2, 3 describe two 'projections' of the three-dimensional graph for $u=u(x,t)$.

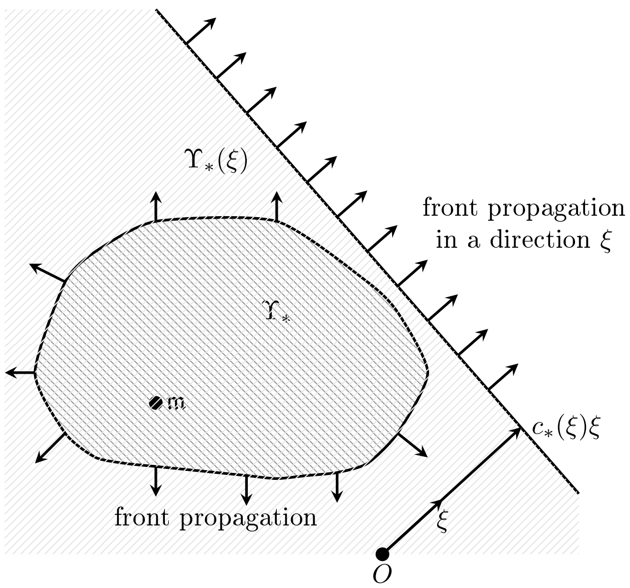

Figure 1: Relationship between the sets $\Upsilon_*(\xi)$ and Υ_*, see [30]

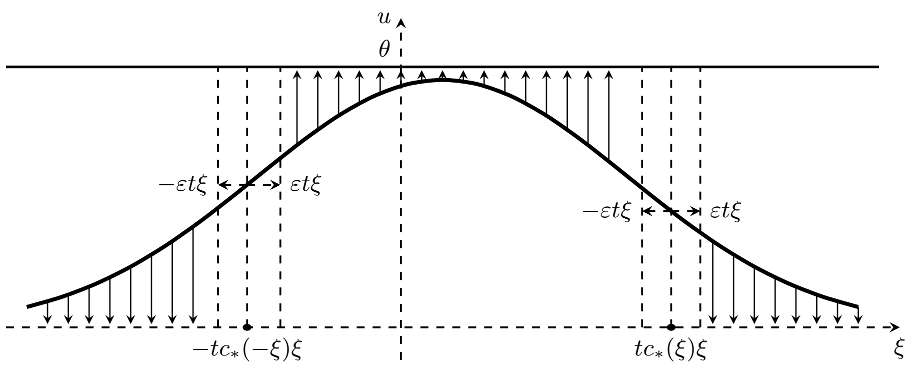

Figure 2: Space-value diagram, see [30]

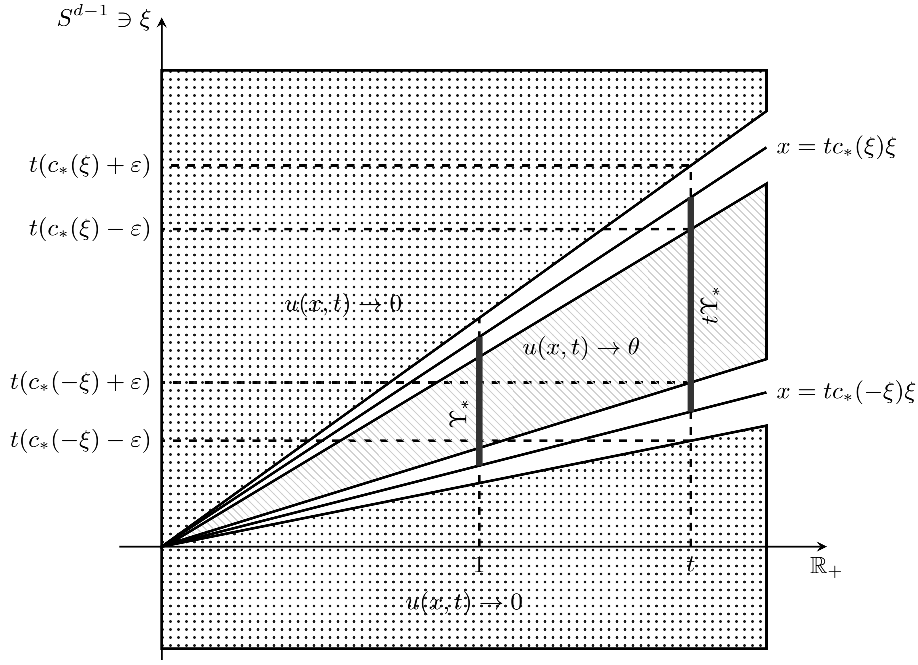

Figure 3: Space-time diagram, see [30]

As it was mentioned above, the front propagation in a direction $\xi \in S^{d-1}$ is deeply related to the minimal speed of traveling wave solutions in the direction ξ. By a (monotone) traveling wave solution to (1.1) in the fixed direction $\xi \in S^{d-1}$, we will understand a solution of the form

$$\begin{gathered} u(x,t) = \psi(x\cdot\xi - ct), \quad t \geq 0, \text{ a.a. } x \in \mathbb{R}^d, \\ \psi(-\infty) = \theta, \qquad \psi(+\infty) = 0, \end{gathered} \tag{1.16}$$

where $c \in \mathbb{R}$ is called the speed of the wave and a decreasing and right-continuous function ψ is called the profile of the wave.

Theorem 1.6 (cf. [13, Theorems 1.1–1.3]). *In conditions and notations of Theorem 1.3, the following statements hold.*

1. *For any $c \geq c_*(\xi)$, there exists a profile $\psi = \psi_c$, such that* (1.16) *defines a solution to* (1.1)*; and for any $c < c_*(\xi)$ a monotone traveling wave solution to* (1.1) *of the form* (1.16) *does not exist.*

2. *The abscissa of a profile $\psi_{*,\xi}$ corresponding to the traveling wave with the minimal speed $c_*(\xi)$ coincides with $\lambda_*(\xi)$, cf.* (1.13)*, namely,*

$$\sup\left\{\lambda > 0 \;\middle|\; \int_{\mathbb{R}} \psi_{*,\xi}(s)e^{\lambda s}\, ds < \infty\right\} = \lambda_*(\xi).$$

Note also that, under some additional technical assumptions, see [13, Theorem 1.3], the profile ψ_c corresponding to a speed $c \geq c_*(\xi)$, $c \neq 0$ is unique (up to a shift). The traveling wave with speed $c_*(\xi)$ is asymptitically stable (see examples in [31]).

On the other hand, if the condition (A3$_\xi$) fails for all $\xi \in S^{d-1}$, then traveling waves do not exist at all.

Proposition 1.7. *Let the conditions* (A1)*,* (A2)*,* (A4)*–*(A6) *hold. Suppose that* (1.10) *also holds. Then there does not exist a traveling wave solution of the form* (1.16) *to the equation* (1.1)*.*

The present paper is a continuation of [12, 13]. They all are based on our unpublished preprint and thesis [30].

For the case of the local nonlinearity in (1.1), when $\varkappa_{n\ell} = 0$, the equation (1.1) was considered, in particular, in [1, 5, 6, 17, 19, 26, 27, 29, 35, 37]. For a nonlocal nonlinearity and, especially, for the case $\varkappa_\ell = 0$ in (1.1), see e.g. [7, 9–11, 14, 16, 25, 36]. For details, see the introduction to [12] and also the comments above.

For the case $\varkappa_\ell = 0$, $\varkappa^+ = \varkappa^- = \varkappa_{n\ell}$, $a^+(x) = a^-(x)$ for $x \in \mathbb{R}^d$, the result similar to Theorem 1.1 can be found in [25], where the viscosity solution technique has been used. If, additionally, $d = 1$ and the kernels $a^\pm$ decay faster than any exponential function, one can refer also to [33]. For the case $\varkappa_{n\ell} = 0$, see also [27].

The paper is organised as follows. In Section 2, we describe the properies of the semi-flow generated by the equation (1.1) and connect Weinberger's scheme [32] with Theorem 1.6. In Section 3, we study the propagation of a solution to (1.1) in a fixed direction and prove the second item of Theorem 1.3. In Subsection 4.1, we find sufficient conditions that Υ_* is a compact and has a non-empty interior, and we prove the first item of Theorem 1.1. In Subsection 4.2, we extend Weinberger's scheme from discrete to continuous time and prove (Proposition 4.12) the convergence (1.8) under additional assumption on the initial condition. Finally, using the hair-trigger effect proved early in [15], we get rid on the latter restriction and prove the second item of Theorem 1.1.

2 Technical tools

2.1 Properties of semi-flow

By a solution to (1.1) on $[0, T)$, $T \leq \infty$, we will understand the so-called classical solution, that is a continuous mapping from $[0, T)$ to the space $E := L^\infty(\mathbb{R}^d)$

which is continuously differentiable (in the sense of the esssup-norm in E) in $t \in (0,T)$, and satisfies (1.1). We denote by $\mathcal{X}_\infty$ the vector space of all continuous mappings from $\mathbb{R}_+$ to E.

By [15, Theorem 2.2], for any $0 \le u_0 \in E$ and for any $T > 0$, there exists a unique classical solution u to (1.1) on $[0,T)$. In particular, $u \in \mathcal{X}_\infty$ is a unique classical solution to (1.1) on $\mathbb{R}_+ := [0,\infty)$.

Moreover, by [15], if u_0 belongs to either of spaces $C_b(\mathbb{R}^d)$ or $C_{ub}(\mathbb{R}^d)$ of bounded continuous or, respectively, bounded uniformly continuous functions on $\mathbb{R}^d$ with sup-norm, then $u(\cdot,t)$ belongs to the same space for all $t > 0$; cf. (Q1) in Theorem 2.1 below.

For any $t \ge 0$ and $0 \le f \in E = L^\infty(\mathbb{R}^d)$, we define the continuous semi-flow (see [12] for details) as follows

$$(Q_t f)(x) := u(x,t), \qquad \text{a.a. } x \in \mathbb{R}^d, \tag{2.1}$$

where $u(x,t)$ is the solution to (1.1) with the initial condition $u(x,0) = f(x)$. We denote,

$$E_\theta^+ := \{u \in E \mid 0 \le u \le \theta\}.$$

Here and in the sequel, we will understand all inequalities between functions from E almost everywhere only.

Theorem 2.1 ([12, Theorem 1.5]; see also [15, Proposition 5.4])**.** *Let* (A1)–(A2) *hold. Let $(Q_t)_{t\ge0}$ be the semi-flow* (2.1) *on the cone $\{0 \le f \in E\}$. Then, for each $t > 0$, $Q = Q_t$ satisfies the following properties:*

(Q1) *Q maps each of sets E_θ^+, $E_\theta^+ \cap C_b(\mathbb{R}^d)$, $E_\theta^+ \cap C_{ub}(\mathbb{R}^d)$ into itself;*

(Q2) *let T_y, $y \in \mathbb{R}^d$, be a translation operator, given by*

$$(T_y f)(x) = f(x-y), \quad x \in \mathbb{R}^d,$$

then

$$(QT_y f)(x) = (T_y Q f)(x), \quad x,y \in \mathbb{R}^d, \ f \in E_\theta^+;$$

(Q3) *$Q0 = 0$, $Q\theta = \theta$, and $Qr > r$, for any constant $r \in (0,\theta)$;*

(Q4) *if $f,g \in E_\theta^+$, $f \le g$, then $Qf \le Qg$;*

(Q5) *if $f_n, f \in E_\theta^+$, $f_n \overset{\text{loc}}{\Longrightarrow} f$, then $(Qf_n)(x) \to (Qf)(x)$ for (a.a.) $x \in \mathbb{R}^d$;*

(Q6) *if $d = 1$, then $Q : \mathcal{M}_\theta(\mathbb{R}) \to \mathcal{M}_\theta(\mathbb{R})$.*

Here and below $\overset{\text{loc}}{\Longrightarrow}$ denotes the locally uniform convergence of functions on $\mathbb{R}^d$ (in other words, $f_n 1\!\!1_\Lambda$ converge to $f 1\!\!1_\Lambda$ in E, for each compact $\Lambda \subset \mathbb{R}^d$), and $\mathcal{M}_\theta(\mathbb{R})$ denotes the set of all decreasing and right-continuous functions $f : \mathbb{R} \to [0,\theta]$.

For each $0 \le T_1 < T_2 < \infty$, let $\mathcal{X}_{T_1,T_2}$ denote the Banach space of all continuous mappings from $[T_1,T_2]$ to E with the norm

$$\|u\|_{T_1,T_2} := \sup_{t\in[T_1,T_2]} \|u(\cdot,t)\|_E.$$

For any $T>0$, we set also $\mathcal{X}_T := \mathcal{X}_{0,T}$ and consider the subset $\mathcal{U}_T \subset \mathcal{X}_T$ of all mappings which are continuously differentiable on $(0,T]$. Here and below, we consider the left derivative at $t=T$ only.

The property (Q4) gives the comparison principle for solutions to (1.1). To formulate a more general result needed for the sequel, consider, for each $T>0$ and $u\in\mathcal{U}_T$,

$$(\mathcal{F}u)(x,t) := \frac{\partial u}{\partial t}(x,t) - \varkappa^+(a^+ * u)(x,t) + mu(x,t) + u(x,t)\big(Gu\big)(x,t)$$

for all $t\in(0,T]$ and a.a. $x\in\mathbb{R}^d$.

Proposition 2.2 ([12, Proposition 2.8], cf. [15, Theorem 2.3])**.** *Let* (A1)–(A2) *hold. Let $T>0$ be fixed and $u_1,u_2\in\mathcal{U}_T$ be such that, for all $t\in(0,T]$, $x\in\mathbb{R}^d$,*

$$\begin{gathered}(\mathcal{F}u_1)(x,t) \leq (\mathcal{F}u_2)(x,t),\\ 0\leq u_1(x,t)\leq\theta, \qquad 0\leq u_2(x,t)\leq\theta,\\ 0\leq u_1(x,0)\leq u_2(x,0)\leq\theta.\end{gathered}$$

Then, for all $t\in[0,T]$, $x\in\mathbb{R}^d$,

$$0\leq u_1(x,t)\leq u_2(x,t)\leq\theta.$$

We will need also a weaker form of (Q5) under weaker assumptions.

Proposition 2.3. *Let* (A1), (A2) *hold. Let $(Q_t)_{t\geq0}$ be the semi-flow* (2.1) *on the cone $\{0\leq f\in E\}$. Let $T>0$ be fixed. Consider a sequence of functions $u_n\in\mathcal{X}_T$ which are solutions to* (1.1) *with uniformly bounded initial conditions: $u_n(\cdot,0)\in E_\theta^+$, $n\in\mathbb{N}$. Let $u\in\mathcal{X}_T$ be a solution to* (1.1) *with initial condition $u(\cdot,0)$ such that $u_n(x,0)\to u(x,0)$, for a.a. $x\in\mathbb{R}^d$. Then $u_n(x,t)\to u(x,t)$, for a.a. $x\in\mathbb{R}^d$, uniformly in $t\in[0,T]$.*

Proof. Clearly, $u_n(\cdot,0)\in E_\theta^+$ implies $u(\cdot,0)\in E_\theta^+$. By (Q1), $u_n(\cdot,t),u(\cdot,t)\in E_\theta^+$, $n\in\mathbb{N}$, for any $t\geq0$. We define, for any $n\in\mathbb{N}$,

$$\overline{u}_n(x,0) := \max\{u_n(x,0),u(x,0)\}, \qquad \underline{u}_n(x,0) := \min\{u_n(x,0),u(x,0)\}.$$

Then, clearly, $0\leq\underline{u}_n(x,0)\leq u(x,0)\leq\overline{u}_n(x,0)\leq\theta$, $n\in\mathbb{N}$, a.a. $x\in\mathbb{R}^d$. Hence the corresponding solutions $\overline{u}_n(x,t)$, $\underline{u}_n(x,t)$ to (1.1) belongs to E_θ^+ as well. By (Q4), one has $\underline{u}_n(x,t)\leq u(x,t)\leq\overline{u}_n(x,t)$, $n\in\mathbb{N}$, $t\in[0,T]$, a.a. $x\in\mathbb{R}^d$. In the same way, one gets $\underline{u}_n(x,t)\leq u_n(x,t)\leq\overline{u}_n(x,t)$ a.e. on $\mathbb{R}^d\times[0,T]$. Therefore, it is enough to prove that $\overline{u}_n$ and $\underline{u}_n$ converge a.e. to u

Prove that $\overline{u}_n(x,t)\to u(x,t)$ for a.a. $x\in\mathbb{R}^d$ uniformly in $t\in[0,T]$. For any $n\in\mathbb{N}$, the function $h_n(\cdot,t)=\overline{u}_n(\cdot,t)-u(\cdot,t)\in E_\theta^+$, $t\geq0$, satisfies the equation $\frac{\partial}{\partial t}h_n = P_nh_n$ with $h_{n,0}(x):=h_n(x,0)=\overline{u}_n(x,0)-u(x,0)\geq0$, a.a. $x\in\mathbb{R}^d$, where, for any $0\leq h\in\mathcal{X}_T$,

$$P_nh := -mh + \varkappa^+(a^+ * h) - \varkappa_{n\ell}h(a^- * \overline{u}_n) - \varkappa_{n\ell}u(a^- * h) - \varkappa_\ell h(u+\overline{u}_n).$$

For any u_n and u, P_n is a bounded linear operator on E, therefore, $h_n(x,t)=(e^{tP_n}h_{n,0})(x)$, a.a. $x\in\mathbb{R}^d$, $t\in[0,T]$. Since $u\geq0$, one has that, for any $0\leq h\in\mathcal{X}_T$, $(P_nh)(x,t)\leq(Ph)(x,t)$, a.a. $x\in\mathbb{R}^d$, $t\in[0,T]$, where a bounded linear operator P is given on E by

$$Ph := \varkappa^+(a^+ * h) - \varkappa_{n\ell}u(a^- * h) - \varkappa_\ell uh.$$

Next, the series expansions for e^{tP_n} and e^{tP} converge in the topology of norms of operator on the space E. Then, for any $n \in \mathbb{N}$, $t \in [0,T]$ and a.a. $x \in \mathbb{R}^d$,

$$h_n(x,t) = (e^{tP_n} h_{n,0})(x) \le (e^{TP} h_{n,0})(x) = \sum_{m=0}^{\infty} \frac{T^m}{m!} P^m h_{n,0}(x), \tag{2.2}$$

and, moreover, for any $\varepsilon > 0$ and a.a. $x \in \mathbb{R}^d$, one can find $M = M(\varepsilon, x) \in \mathbb{N}$, such that we get from (2.2) that, for $t \in [0,T]$ and a.a. $x \in \mathbb{R}^d$,

$$h_n(x,t) \le \sum_{m=0}^{M} \frac{T^m}{m!} P^m h_{n,0}(x) + \varepsilon\theta, \tag{2.3}$$

as $h_{n,0} \in E_\theta^+$, $n \in \mathbb{N}$. Finally, the assumptions of the statement yield that $h_{n,0}(x) \to 0$ for a.a. $x \in \mathbb{R}^d$. Then, by (2.3) and [12, Lemma 2.2], $h_n(x,t) \to 0$ for a.a. $x \in \mathbb{R}^d$ uniformly in $t \in [0,T]$. Hence, $\overline{u}_n(x,t) \to u(x,t)$ for a.a. $x \in \mathbb{R}^d$ uniformly on $[0,T]$. The convergence for $\underline{u}_n(x,t)$ may be proved by an analogy. □

2.2 Around Weinberger's scheme

We will follow the abstract scheme proposed in [32]. Let (A1)–(A2) hold. We introduce the following notation, cf. (Q1) of Theorem 2.1,

$$C_\theta := E_\theta^+ \cap C_b(\mathbb{R}^d). \tag{2.4}$$

Consider the set N_θ of all non-increasing functions $\varphi \in C(\mathbb{R})$, such that $\varphi(s) = 0$, $s \ge 0$, and

$$\varphi(-\infty) := \lim_{s\to-\infty} \varphi(s) \in (0,\theta).$$

It is easily seen that $N_\theta \subset C_\theta$.

For arbitrary $s \in \mathbb{R}$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, we define the mapping $V_{s,c,\xi} : L^\infty(\mathbb{R}) \to E$ as follows

$$(V_{s,c,\xi} f)(x) := f(x\cdot\xi + s + c), \quad x \in \mathbb{R}^d. \tag{2.5}$$

Fix an arbitrary $\varphi \in N_\theta$. For $T > 0$, $c \in \mathbb{R}$, $\xi \in S^{d-1}$, consider the mapping $R_{T,c,\xi} :\ L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, given by

$$(R_{T,c,\xi} f)(s) := \max\bigl\{\varphi(s), (Q_T(V_{s,c,\xi} f))(0)\bigr\}, \quad s \in \mathbb{R}, \tag{2.6}$$

where Q_T is given by (2.1). Consider now the following sequence of functions

$$f_{n+1}(s) := (R_{T,c,\xi} f_n)(s), \quad f_0(s) := \varphi(s), \qquad s \in \mathbb{R}, n \in \mathbb{N}\cup\{0\}. \tag{2.7}$$

By Theorem 2.1 and [32, Lemma 5.1], $\varphi \in C_\theta$ implies $f_n \in C_\theta$ and $f_{n+1}(s) \ge f_n(s)$, $s \in \mathbb{R}$, $n \in \mathbb{N}$; hence one can define the following limit

$$f_{T,c,\xi}(s) := \lim_{n\to\infty} f_n(s), \quad s \in \mathbb{R}. \tag{2.8}$$

Also, by [32, Lemma 5.1], for fixed $\xi \in S^{d-1}$, $T > 0$, $n \in \mathbb{N}$, the functions $f_n(s)$ and $f_{T,c,\xi}(s)$ are nonincreasing in s and in c; moreover, $f_{T,c,\xi}(s)$ is a lower semicontinuous function of s, c, ξ, as a result, this function is continuous from the right in s and in c. Note also, that $0 \le f_{T,c,\xi} \le \theta$. Then, for any c, ξ, one can define the limiting value

$$f_{T,c,\xi}(\infty) := \lim_{s\to\infty} f_{T,c,\xi}(s).$$

Next, for any $T > 0$, $\xi \in S^{d-1}$, we define

$$c_T^*(\xi) := \sup\{c \mid f_{T,c,\xi}(\infty) = \theta\} \in \mathbb{R} \cup \{-\infty, \infty\}, \tag{2.9}$$

where, as usual, $\sup \emptyset := -\infty$. By [32, Propositions 5.1, 5.2], one has

$$f_{T,c,\xi}(\infty) = \begin{cases} \theta, & c < c_T^*(\xi), \\ 0, & c \ge c_T^*(\xi), \end{cases} \tag{2.10}$$

cf. also [32, Lemma 5.5]; moreover, $c_T^*(\xi)$ is a lower semicontinuous function of $\xi \in S^{d-1}$. It is crucial that, by [32, Lemma 5.4], neither $f_{T,c,\xi}(\infty)$ nor $c_T^*(\xi)$ depends on the choice of $\varphi \in N_\theta$. Note that the monotonicity of $f_{T,c,\xi}(s)$ in s and (2.10) imply that, for $c < c_T^*(\xi)$, $f_{T,c,\xi}(s) = \theta$, $s \in \mathbb{R}$.

Define now the following set, cf. (2.9),

$$\Upsilon_{T,\xi} = \bigl\{x \in \mathbb{R}^d \mid x \cdot \xi \le c_T^*(\xi)\bigr\}, \quad \xi \in S^{d-1}, T > 0. \tag{2.11}$$

Clearly, the set $\Upsilon_{T,\xi}$ is convex and closed.

Recall that, under (A1)–(A2), $c_*(\xi)$, $\xi \in S^{d-1}$, is given by (1.4).

Proposition 2.4. *Let* (A1)–(A2) *hold. Then, for any $\xi \in S^{d-1}$, $c_*(\xi) < \infty$ if and only if $c_T^*(\xi) < \infty$ for all $T > 0$, and*

$$c_T^*(\xi) = T c_*(\xi), \quad T > 0. \tag{2.12}$$

As a result, cf. (1.11), (2.11),

$$\Upsilon_{T,\xi} = T\Upsilon_{1,\xi} = T\Upsilon_*(\xi), \quad T > 0. \tag{2.13}$$

Proof. Let $T > 0$ and $c_T^*(\xi) < \infty$. Take any $c \in \mathbb{R}$ with $cT \ge c_T^*(\xi)$. Then, by (2.10), $f_{T,cT,\xi} \not\equiv \theta$. By (2.6), (2.7), one has

$$f_{n+1}(s) \ge (Q_T(V_{s,cT,\xi} f_n))(0), \quad s \in \mathbb{R}. \tag{2.14}$$

Since $f_n(s)$ is nonincreasing in s, one gets, by (2.5), that, for a fixed $x \in \mathbb{R}^d$, the function $(V_{s,cT,\xi} f_n)(x)$ is also nonincreasing in s. Next, by (2.5), (2.8) and Propositions 2.3,

$$(Q_T(V_{s,cT,\xi} f_n))(x) \to (Q_T(V_{s,cT,\xi} f_{T,cT,\xi}))(x), \ \text{a.a. } x \in \mathbb{R}^d. \tag{2.15}$$

Note that, by (2.5) and [12, Proposition 3.3],

$$(Q_T(V_{s,cT,\xi} f_{T,cT,\xi}))(x) = \phi(x \cdot \xi, T), \tag{2.16}$$

where $\phi(\tau, t)$, $\tau \in \mathbb{R}$, $t \in \mathbb{R}_+$ solves

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(s,t) = \varkappa^+(\check{a}^+ * \phi)(s,t) - m\phi(s,t) - \varkappa_\ell \phi^2(s,t) \\ \qquad\qquad - \varkappa_{n\ell}\phi(s,t)(\check{a}^- * \phi)(s,t), \qquad t > 0, \ \text{a.a. } s \in \mathbb{R}, \\ \phi(s,0) = \psi(s), \qquad \text{a.a. } s \in \mathbb{R}. \end{cases} \tag{2.17}$$

with $\psi(\tau) = f_{T,cT,\xi}(\tau + s + cT)$ (note that s is a parameter now, cf. (2.17)), and

$$\breve{a}^{\pm}(s) := \int_{\{\xi\}^{\perp}} a^{\pm}(s\xi + \eta)\, d\eta, \quad s \in \mathbb{R},$$

where $\{\xi\}^{\perp} := \{x \in \mathbb{R}^d \mid x \cdot \xi = 0\}$.

On the other hand, the evident equality

$$(V_{s,cT,\xi} f_{T,cT,\xi})(x + \tau\xi) = f_{T,cT,\xi}(x \cdot \xi + \tau + s + cT), \quad \tau \in \mathbb{R}$$

shows that the function $V_{s,cT,\xi} f_{T,cT,\xi}$ is a decreasing function on $\mathbb{R}^d$ along the $\xi \in S^{d-1}$ as $f_{T,cT,\xi}$ is a decreasing function on $\mathbb{R}$. Then, by [12, Proposition 2.7] and (2.16), the function $\mathbb{R}^d \ni x \mapsto \phi(x \cdot \xi, T) \in [0, \theta]$ is decreasing along the ξ as well, i.e.

$$\phi(x \cdot \xi + \tau, T) = \phi((x + \tau\xi) \cdot \xi, T) \leq \phi(x \cdot \xi, T), \quad \tau \geq 0.$$

As a result, the function $\phi(s, T)$ is monotone (almost everywhere) in s. Since $f_{T,cT,\xi}(s)$ was continuous from the right in s, one gets from (2.14), (2.15), that

$$f_{T,cT,\xi}(s) \geq (\tilde{Q}_T f_{T,cT,\xi})(s + cT),$$

where $\widetilde{Q}_t : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is defined as follows: $\widetilde{Q}_t \psi(s) = \phi(s,t)$, $s \in \mathbb{R}$, where $\phi : \mathbb{R} \times \mathbb{R}_+ \to [0, \theta]$ solves (2.17) with $0 \leq \psi \in L^\infty_+(\mathbb{R})$. Since $f_{T,cT,\xi} \not\equiv \theta$, one has that, by [35, Theorem 5] (cf. the proof of [12, Theorem 1.1]), there exists a traveling wave profile with the speed c. By Theorem 1.6, we have that $c \geq c_*(\xi)$, and hence $Tc_*(\xi) \leq c^*_T(\xi) < \infty$.

Let now $T > 0$ and $c_*(\xi) < \infty$. Take any $c \geq c_*(\xi)$ and consider, by Theorem 1.6, a traveling wave in a direction $\xi \in S^{d-1}$, with a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ and the speed c. Then, by (2.5) and (1.16),

$$(Q_T(V_{s,cT,\xi}\psi))(x) = \psi((x \cdot \xi - cT) + s + cT) = \psi(x \cdot \xi + s).$$

Choose $\varphi \in N_\theta$ such that $\varphi(s) \leq \psi(s)$, $s \in \mathbb{R}$ (recall that all constructions are independent on the choice of φ). Then, one gets from (2.6) and (Q4) of Theorem 2.1, that

$$(R_{T,cT,\xi}\varphi)(s) \leq (R_{T,cT,\xi}\psi)(s) = \psi(s), \quad s \in \mathbb{R}.$$

Then, by (2.7) and (2.8), $f_{T,cT,\xi}(s) \leq \psi(s)$, $s \in \mathbb{R}$, and thus (2.10) implies $cT \geq c^*_T(\xi)$; as a result, $c^*_T(\xi) \leq Tc_*(\xi) < \infty$, that fulfills the statement. □

A developement of Weinberger's scheme crucial for the sequel is the so-called *hair-trigger effect*. We have proved it for a generalisation of (1.1) in [15]. It is straightforward to check, cf. [12, Subsection 2.1], that, in our settings, the result can be read as follows.

Theorem 2.5 (cf. [15, Theorem 2.5])**.** *Let the conditions* (A1)*,* (A2)*,* (A4)*–*(A6) *hold. Let $u_0 \in E^+_\theta$ be such that there exist $x_0 \in \mathbb{R}^d$, $\eta > 0$, $r > 0$, with $u_0 \geq \eta$, for a.a. $x \in B_r(x_0)$. Let $u \in \mathcal{X}_\infty$ be the corresponding classical solution to* (1.1) *on $\mathbb{R}_+$. Then, for $\mathfrak{m}$ defined by* (1.15) *and any compact set $K \subset \mathbb{R}^d$,*

$$\lim_{t\to\infty} \operatorname*{essinf}_{x \in K} u(x + t\mathfrak{m}, t) = \theta.$$

In particular, if $\mathfrak{m} = 0 \in \mathbb{R}^d$, then the solution to (1.1) converges to θ locally uniformly. Our main aim in the rest of the paper is to show that the zone where the solution to (1.1) becomes arbitrary close to θ (as time grows to ∞) can be chosen expanding to $\mathbb{R}^d$ linearly in time, cf. (4.7) below.

3 Long-time behavior in a direction

In this Section, we are going to prove the second item of Theorem 1.3. We start with the following simple observation. Let $0 \le u_0 \in E$ be an initial condition to (1.1) and $u = u(x,t) \ge 0$ be the corresponding solution. Then, by Duhamel's principle, $u(x,t) \le w(x,t)$, $x \in \mathbb{R}^d$, $t > 0$, where $w(x,t)$ is the solution to the linear equation

$$\frac{\partial w}{\partial t}(x,t) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x-y)w(y,t)dy - mw(x,t) \tag{3.1}$$

with the same initial condition $w(x,0) = u_0(x)$, $x \in \mathbb{R}^d$. We will find now an appropriate upper estimate for the solution to (3.1).

To this end, for any $\xi \in S^{d-1}$ and $\lambda > 0$, consider the following set of bounded functions on $\mathbb{R}^d$:

$$E_{\lambda,\xi}(\mathbb{R}^d) := \bigl\{ f \in E \bigm| \|f\|_{\lambda,\xi} := \operatorname*{esssup}_{x\in\mathbb{R}^d} |f(x)|e^{\lambda x\cdot\xi} < \infty \bigr\}. \tag{3.2}$$

Evidently, for $f \in E$,

$$\operatorname*{esssup}_{x\in\mathbb{R}^d} |f(x)|e^{\lambda x\cdot\xi} < \infty \quad \text{if and only if} \quad \operatorname*{esssup}_{x\cdot\xi\ge 0} |f(x)|e^{\lambda x\cdot\xi} < \infty,$$

therefore,

$$E_{\lambda,\xi}(\mathbb{R}^d) \subset E_{\lambda',\xi}(\mathbb{R}^d), \quad \lambda > \lambda' > 0,\ \xi \in S^{d-1}.$$

Proposition 3.1. *Let $\xi \in S^{d-1}$ and $\lambda > 0$ be fixed and suppose that* (A3$_\xi$) *holds with $\mu = \lambda$. Let $0 \le u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$ and let $w = w(x,t)$ be the solution to* (3.1) *with the initial condition $w(x,0) = u_0(x)$, $x \in \mathbb{R}^d$. Then*

$$\|w(\cdot,t)\|_{\lambda,\xi} \le \|u_0\|_{\lambda,\xi} e^{pt}, \quad t \ge 0, \tag{3.3}$$

where

$$p = p(\xi,\lambda) = \varkappa^+ \int_{\mathbb{R}^d} a^+(x)e^{\lambda x\cdot\xi}\, dx - m \in \mathbb{R}. \tag{3.4}$$

Proof. First, we note that, for any $a \in L^1(\mathbb{R}^d)$, $f \in E_{\lambda,\xi}(\mathbb{R}^d)$

$$\begin{aligned} \bigl|(a*f)(x)e^{\lambda x\cdot\xi}\bigr| &\le \int_{\mathbb{R}^d} |a(x-y)|e^{\lambda(x-y)\cdot\xi}|f(y)|e^{\lambda y\cdot\xi}\, dy \\ &\le \|f\|_{\lambda,\xi} \int_{\mathbb{R}^d} |a(y)|e^{\lambda y\cdot\xi}\, dy. \end{aligned} \tag{3.5}$$

Applying (3.5) to $a = a^+ \in L^1(\mathbb{R}^d)$ and $f = u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$, and using the notation (1.3), we will get

$$\|a^+ * u_0\|_{\lambda,\xi} \le \mathfrak{a}_\xi(\lambda)\|u_0\|_{\lambda,\xi}.$$

Iteratively applying (3.5) to $a = a^+$ and $f = a^{+,*(n-1)} * u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$, $n \ge 2$, where $a^{+,*(n-1)} := a^+ * \ldots * a^+$ (the convolution is taken $n-2$ times), we obtain

$$\|a^{+,*n} * u_0\|_{\lambda,\xi} \le \bigl(\mathfrak{a}_\xi(\lambda)\bigr)^n \|u_0\|_{\lambda,\xi}.$$

Since the operator in the right hand side of (3.1) is bounded in E, we have an explicit representation for the solution to (3.1), namely,

$$w(x,t)=e^{-mt}u_0(x)+e^{-mt}\sum_{n=1}^{\infty}\frac{(\varkappa^+t)^n}{n!}\bigl(a^{+,*n}*u_0\bigr)(x),\quad x\in\mathbb{R}^d,\ t\geq 0.$$

As a result, we obtain

$$\|w(\cdot,t)\|_{\lambda,\xi}\leq e^{-mt}\|u_0\|_{\lambda,\xi}+e^{-mt}\sum_{n=1}^{\infty}\frac{(\varkappa^+t)^n}{n!}\bigl(\mathfrak{a}_\xi(\lambda)\bigr)^n\|u_0\|_{\lambda,\xi},$$

that is just equivalent to (3.3)–(3.4). □

Remark 3.2. It is straightforward to check, cf. [13, Lemma 2.1], that the statement of Proposition 3.1 remains true if (A3$_\xi$) holds for some $\mu>\lambda$, provided that we assume, additionally, (A4).

We can prove now the second item of Theorem 1.3.

Proof. Let $p_*:=p(\xi,\lambda_*)$ be given by (3.4). Let $w=w(x,t)$ be the solution to (3.1) with the initial condition $w(x,0)=u_0(x)$, $x\in\mathbb{R}^d$. By (3.3), (3.2), one has

$$0\leq u(x,t)\leq w(x,t)\leq\|u_0\|_{\lambda_*,\xi}\exp\bigl\{p_*t-\lambda_*x\cdot\xi\bigr\},\quad \text{a.a. } x\in\mathbb{R}^d. \tag{3.6}$$

Next, by (2.11) and Proposition 2.4, for any $t>0$ and for all $x\in\mathbb{R}^d\setminus t\mathscr{O}_\xi$, one has $x\cdot\xi\geq tc_1^*(\xi)+t\delta=tc_*(\xi)+t\delta$. Then, by (1.13),

$$\begin{aligned}\inf_{x\notin t\mathscr{O}_\xi}(\lambda_*x\cdot\xi)&\geq t\lambda_*c_*(\xi)+t\lambda_*\delta\\&=t\Bigl(\varkappa^+\int_{\mathbb{R}^d}a^+(x)e^{\lambda_*x\cdot\xi}\,dx-m\Bigr)+t\lambda_*\delta=tp_*+t\lambda_*\delta.\end{aligned}$$

Therefore, (3.6) implies the statement. □

Remark 3.3. The assumption $u_0\in E_{\lambda_*,\xi}(\mathbb{R}^d)$ is close, in some sense, to the weakest possible assumption on an initial condition $u_0\in E_\theta^+$ for the equation (1.1) to have

$$\lim_{t\to\infty}\operatorname*{esssup}_{x\notin t\mathscr{O}_\xi}u(x,t)=0, \tag{3.7}$$

for an arbitrary open set $\mathscr{O}_\xi\supset\Upsilon_{1,\xi}$, where $\Upsilon_{1,\xi}$ is defined by (2.11). Indeed, take any λ_1,λ with $0<\lambda_1<\lambda<\lambda_*=\lambda_*(\xi)$. By Theorem 1.6, there exists a traveling wave solution to (1.1) with a profile $\psi_1\in\mathcal{M}_\theta(\mathbb{R})$ such that $\lambda_0(\psi_1)=\lambda_1$. By [13, Theorem 1.3] (with $j=1$ as $\lambda_1<\lambda_*$) we have that $\psi_1(t)\sim De^{-\lambda_1 t}$, $t\to\infty$. It is easily seen that one can choose a function $\varphi\in\mathcal{M}_\theta(\mathbb{R})\cap C(\mathbb{R})$ such that there exist $p>0$, $T>0$, such that $\varphi(t)\geq\psi_1(t)$, $t\in\mathbb{R}$ and $\varphi(t)=pe^{-\lambda t}$, $t>T$. Take now $u_0(x)=\varphi(x\cdot\xi)$, $x\in\mathbb{R}^d$. We have $u_0\in E_{\lambda,\xi}(\mathbb{R}^d)\setminus E_{\lambda_*,\xi}(\mathbb{R}^d)$. Then, by [12, Proposition 3.3], the corresponding solution has the form $u(x,t)=\phi(x\cdot\xi,t)$. By Proposition 2.2 applied to the equation (2.17), $\phi(s,t)\geq\psi_1(s-c_1t)$, $s\in\mathbb{R}$, $t\geq 0$, where $c_1=\lambda_1^{-1}(\varkappa^+\mathfrak{a}_\xi(\lambda_1)-m)>c_*(\xi)$, cf. [13, formula (1.13)].

Take $c \in (c_*(\xi), c_1)$ and consider an open set $\mathscr{O}_\xi := \{x \in \mathbb{R}^d \mid x \cdot \xi < c\}$, then $\Upsilon_{1,\xi} \subset \mathscr{O}_\xi \subset \{x \in \mathbb{R}^d \mid x \cdot \xi \le c_1\} =: A_1$. One has

$$\begin{aligned}\sup_{x \notin t\mathscr{O}_\xi} u(x,t) &\ge \sup_{x \in tA_1 \setminus t\mathscr{O}_\xi} \phi(x \cdot \xi, t) \\ &\ge \sup_{ct < s \le c_1 t} \psi_1(s - c_1 t) = \psi_1(ct - c_1 t) > \psi_1(0),\end{aligned}$$

as $c < c_1$ and ψ_1 is decreasing. As a result, (3.7) does not hold.

On the other hand, if $\psi_* \in \mathcal{M}_\theta(\mathbb{R})$ is a profile with the minimal speed $c_*(\xi) \neq 0$ and if $j = 2$, cf. [13, Proposition 3.1], then $u_0(x) := \psi_*(x \cdot \xi)$ does not belong to the space $E_{\lambda_*,\xi}(\mathbb{R}^d)$, and the arguments above do not contradict (3.7) anymore. In the next remark, we consider this case in more details.

Remark 3.4. In connection with the previous remark, it is worth noting also that one can easily generalize the second item of Theorem 1.3 in the following way. Let $u_0 \in E_{\lambda,\xi}(\mathbb{R}^d) \cap E_\theta^+$, for some $\lambda \in (0, \lambda_*]$, and let $u \in \widetilde{\mathcal{X}}_\infty$ be the corresponding solution to (1.1). Consider the set $A_{c,\xi} := \{x \in \mathbb{R}^d \mid x \cdot \xi \le c\}$, where $c = \lambda^{-1}(\varkappa^+ \mathfrak{a}_\xi(\lambda) - m)$ cf. [13, formula (1.12)]. Then, for any open set $B_{c,\xi} \supset A_{c,\xi}$ with $\delta_c := \operatorname{dist}(A_{c,\xi}, \mathbb{R}^d \setminus B_{c,\xi}) > 0$, one gets

$$\operatorname*{esssup}_{x \notin tB_{c,\xi}} u(x,t) \le \|u_0\|_{\lambda,\xi} e^{-\lambda \delta_c t}. \tag{3.8}$$

Therefore, if $u_0(x) = \psi_*(x \cdot \xi)$, where ψ_* is as in Remark 3.3 above, then, evidently, $u_0 \in E_{\lambda,\xi}(\mathbb{R}^d)$, for any $\lambda \in (0, \lambda_*)$. Then, for any open $\mathscr{O}_\xi \supset \Upsilon_{1,\xi}$ with $\delta := \operatorname{dist}(\Upsilon_{1,\xi}, \mathbb{R}^d \setminus \mathscr{O}_\xi) > 0$ one can choose, for any $\varepsilon \in (0,1)$, $c_1 = c_*(\xi) + \delta\varepsilon$. By Theorems 1.3 and 1.6, there exists a unique $\lambda_1 = \lambda_1(\varepsilon) \in (0, \lambda_*)$ such that $c_1 = \lambda_1^{-1}(\varkappa^+ \mathfrak{a}_\xi(\lambda_1) - m)$. Then $u_0 \in E_{\lambda_1,\xi}(\mathbb{R}^d)$ and $A_{c_1,\xi} \subset \mathscr{O}_\xi$, i.e. $\mathscr{O}_\xi$ may be considered as a set $B_{c_1,\xi}$, cf. above. As a result, (3.8) gives (1.14), with the constant $\|u_0\|_{\lambda_1,\xi} < \|u_0\|_{\lambda_*,\xi}$, and with $\lambda_* \delta$ replaced by $\lambda_1 \delta(1 - \varepsilon)$. Note that, clearly, $\|u_0\|_{\lambda_1,\xi} \nearrow \|u_0\|_{\lambda_*,\xi}$, $\lambda_1 \nearrow \lambda_*$, $\varepsilon \to 0$.

4 Long-time behavior in different directions

4.1 Convergence to 0

Through this section we will assume that the conditions (A1), (A2), (A4)–(A6) hold. Let the convex closed set Υ_* be given by (1.5). Define, cf. (2.11),

$$\Upsilon_T = \left\{x \in \mathbb{R}^d | x \cdot \xi \le c_T^*(\xi),\ \xi \in S^{d-1}\right\}, \quad T > 0.$$

By (2.11)-(2.13),

$$\Upsilon_T = \bigcap_{\xi \in S^{d-1}} \Upsilon_{T,\xi} = \bigcap_{\xi \in S^{d-1}} T\Upsilon_{1,\xi} = T\Upsilon_1 = T\Upsilon_*, \quad T > 0; \tag{4.1}$$

in particular, $\Upsilon_* = \Upsilon_1$.

Proposition 4.1. *Let* (A1), (A2), (A4)–(A6) *hold. Then, cf.* (1.15), $\mathfrak{m}$ *is an interior point of* Υ_*.

Proof. Firstly, if (A3$_\xi$) fails for all $\xi \in S^{d-1}$ then $\Upsilon_* = \mathbb{R}^d$ and the statement is trivial. Next, for an arbitrary $\xi \in S^{d-1}$ such that (A3$_\xi$) holds, we have, by (1.12) and the inequality in (1.13), that

$$\mathfrak{m} \cdot \xi = \varkappa^+ \int_{\mathbb{R}^d} x \cdot \xi a^+(x)\, dx = \mathfrak{m}_\xi < c_*(\xi).$$

Therefore, cf. (1.11), $\mathfrak{m} \in \Upsilon_*(\xi)$, $\xi \in S^{d-1}$. Next, as it was already mentioned, by [32, Proposition 5.1], the function $c_1^*(\xi)$ is lower-semicontinuous in $\xi \in S^{d-1}$. Therefore, by (2.12), the function $c_*(\xi) - \mathfrak{m}_\xi > 0$ is lower-semicontinuous on the compact S^{d-1}, and hence attains its minimum, which we denote by $d_0 > 0$. As a result, $\mathfrak{m} \cdot \xi < c_*(\xi) - d_0$ for all $\xi \in S^{d-1}$, and therefore, an open ball with center at $\mathfrak{m}$ and radius d_0 belongs to the interior of $\Upsilon_*(\xi)$, for each $\xi \in S^{d-1}$. From this, by (1.5), one gets the statement. □

Proposition 4.2. *Let* (A1), (A2), (A4)–(A7) *hold. Then,* $\Upsilon_* = \Upsilon_1$ *is a compact.*

Proof. First, (A7) implies that (A3$_\xi$) holds for all $\xi \in S^{d-1}$. Then, by Theorem 1.6, $c_*(\xi) < \infty$ for all $\xi \in S^{d-1}$. Next, by (1.12) and Proposition 4.1, for any orthonormal basis $\{e_i \mid 1 \leq i \leq d\} \subset S^{d-1}$, $\mathfrak{m} = \sum\limits_{i=1}^{d} \mathfrak{m}_{e_i} e_i \in \operatorname{int}(\Upsilon_*)$. By Theorem 1.6, $x \in \Upsilon_*$ implies that, for any fixed $\xi \in S^{d-1}$, $x \cdot \xi \leq c_*(\xi)$ and $x \cdot (-\xi) \leq c_*(-\xi)$, i.e.

$$-c_*(-\xi) \leq x \cdot \xi \leq c_*(\xi), \quad x \in \Upsilon_*,\ \xi \in S^{d-1}. \tag{4.2}$$

Then (4.2) implies

$$|x \cdot \xi| \leq \max\bigl\{|c_*(\xi)|, |c_*(-\xi)|\bigr\}, \quad x \in \Upsilon_*,\ \xi \in S^{d-1};$$

in particular, for an orthonormal basis $\{e_i \mid 1 \leq i \leq d\}$ of $\mathbb{R}^d$, one gets

$$|x| \leq \sum_{i=1}^{d} |x \cdot e_i| \leq \sum_{i=1}^{d} \max\bigl\{|c_*(e_i)|, |c_*(-e_i)|\bigr\} =: R < \infty, \quad x \in \Upsilon_*,$$

that fulfills the statement. □

Remark 4.3. Here and in Propositions 4.6, 4.7, the condition (A6) can be weaken to (A8). As a matter of fact, it is enough to assume that (A10$_\xi$) holds for all $\xi \in S^{d-1}$.

Remark 4.4. Since $\int_{x\cdot\xi\leq 0} a^+(x) e^{\lambda x\cdot\xi}\, dx \in [0,1]$, $\xi \in S^{d-1}$, $\lambda > 0$, we have the following observation. If, for some $\xi \in S^{d-1}$, there exist $\mu^\pm > 0$, such that, cf. (1.3), $\mathfrak{a}_{\pm\xi}(\mu^\pm) < \infty$, i.e. if (A3$_\xi$) holds for both ξ and $-\xi$, then, for $\mu = \min\{\mu^+, \mu^-\}$,

$$\begin{aligned}\int_{\mathbb{R}^d} a^+(x) e^{\mu|x\cdot\xi|}\, dx &= \int_{x\cdot\xi\geq 0} a^+(x) e^{\mu x\cdot\xi}\, dx + \int_{x\cdot\xi< 0} a^+(x) e^{-\mu x\cdot\xi}\, dx \\ &\leq \int_{x\cdot\xi\geq 0} a^+(x) e^{\mu^+ x\cdot\xi}\, dx + \int_{x\cdot(-\xi)> 0} a^+(x) e^{\mu^- x\cdot(-\xi)}\, dx < \infty. \end{aligned}\tag{4.3}$$

Let now $\{e_i \mid 1 \le i \le d\}$ be an orthonormal basis in $\mathbb{R}^d$. Let (A3$_\xi$) holds for $2d$ directions $\{\pm e_i \mid 1 \le i \le d\} \subset S^{d-1}$ and let $\mu_i = \min\{\mu(e_i), \mu(-e_i)\}$, $1 \le i \le d$, cf. (4.3). Set $\mu = \frac{1}{d}\min\{\mu_i \mid 1 \le i \le d\}$. Then, by the triangle and Jensen's inequalities and (4.3), one has

$$\begin{aligned}\int_{\mathbb{R}^d} a^+(x)e^{\mu|x|}\,dx &\le \int_{\mathbb{R}^d} a^+(x)\exp\biggl(\sum_{i=1}^d \frac{1}{d}\mu_i|x\cdot e_i|\biggr)\,dx\\ &\le \sum_{i=1}^d \frac{1}{d}\int_{\mathbb{R}^d} a^+(x)e^{\mu_i|x\cdot e_i|}\,dx < \infty.\end{aligned}$$

Therefore, (A7) is equivalent to that (A3$_\xi$) holds for all $\xi \in S^{d-1}$.

Remark 4.5. It is worth noting that, by (1.13), (1.12), the following inequality holds, cf. (4.2),

$$c_*(\xi) + c_*(-\xi) > \mathfrak{m}_\xi + \mathfrak{m}_{-\xi} = 0.$$

The following two Propositions prove the first item of Theorem 1.1.

Proposition 4.6. *Let the conditions* (A1), (A2), (A4)–(A6) *hold and there exists $\xi \in S^{d-1}$, such that* (A3$_\xi$) *holds. Let $u_0 \in E_\theta^+$ be such that* (1.6) *holds for all those $\xi \in S^{d-1}$ where $c_*(\xi) < \infty$. Let $u \in \mathcal{X}_\infty$ be the corresponding classical solution to* (1.1) *on $\mathbb{R}_+$. Then, for any compact set $\mathscr{C} \subset \mathbb{R}^d \setminus \Upsilon_*$, there exist $\nu = \nu(\mathscr{C}) > 0$ and $D = D(u_0, \mathscr{C}) > 0$, such that*

$$\operatorname*{esssup}_{x\in t\mathscr{C}} u(x,t) \le De^{-\nu t}, \quad t > 0.$$

Proof. Since there exists $\xi \in S^{d-1}$, such that (A3$_\xi$) holds, we will get from (1.5), that $\Upsilon_* \neq \mathbb{R}^d$. Therefore,

$$\Upsilon_* = \bigcap_{\substack{\xi\in S^{d-1}:\\ c_*(\xi)<\infty}} \bigl\{x \in \mathbb{R}^d \mid x\cdot\xi \le c_*(\xi)\bigr\}.$$

Then a closed set $\mathscr{C} \subset \mathbb{R}^d \setminus \Upsilon_*$ satisfies

$$\mathscr{C} \subset \bigcup_{\substack{\xi\in S^{d-1}:\\ c_*(\xi)<\infty}} \bigl\{x \in \mathbb{R}^d \mid c_*(\xi) < x\cdot\xi\bigr\}.$$

Since $\mathscr{C}$ is a compact, there exist $K \in \mathbb{N}$ and $\xi_1, \ldots, \xi_K \in S^{d-1}$, such that $c_*(\xi_i) < \infty$, $1 \le i \le K$ and

$$\mathscr{C} \subset \bigcup_{1\le i\le K} \bigl\{x \in \mathbb{R}^d \mid x\cdot\xi_i > c_*(\xi_i)\bigr\}.$$

Therefore,

$$\mathscr{O} := \mathbb{R}^d \setminus \mathscr{C} \supset \bigcap_{1\le i\le K} \Upsilon_*(\xi_i).$$

Clearly, $\mathscr{O}$ is an open subset of $\mathbb{R}^d$ and $\mathscr{O} \supset \Upsilon_*(\xi_i)$ for $1 \le i \le K$. By the assumption on u_0 and the condition $c_*(\xi_i) < \infty$, $1 \le i \le K$, the inequality (1.6) holds for all $\xi = \xi_i$, $1 \le i \le K$.

Since $\Upsilon_*(\xi_i)$ is a closed set and $\mathscr{C}$ is a compact, we have that

$$\nu_i := \lambda_*(\xi_i)\,\mathrm{dist}\,(\Upsilon_*(\xi_i), \mathscr{C}) > 0, \quad 1 \le i \le K.$$

The inequality $c_*(\xi_i) < \infty$ implies that the condition (A3$_\xi$) holds for $\xi = \xi_i$, $1 \le i \le K$. Therefore, by the second item of Theorem 1.3, one gets, for any $1 \le i \le K$,

$$\operatorname*{esssup}_{x \in t\mathscr{C}} u(x,t) = \operatorname*{esssup}_{x \notin t\mathscr{O}} u(x,t) \le \|u_0\|_{\lambda_*(\xi_i),\xi_i} e^{-\nu_i t} \le De^{-\nu t}, \quad t > 0,$$

where $\nu := \min\{\nu_i \mid 1 \le i \le K\}$, $D := \max\{\|u_0\|_{\lambda_*(\xi_i),\xi_i} \mid 1 \le i \le K\}$. $\square$

Proposition 4.7. *In conditions and notations of Proposition 4.6, we assume, additionally, that the set Υ_* is bounded (and hence compact). Then* (1.9) *holds for any closed set $\mathscr{C} \subset \mathbb{R}^d \setminus \Upsilon_*$.*

Proof. Consider the set $\mathcal{M}$ of all subsets from $\mathbb{R}^d$ of the following form:

$$M = M_{\varepsilon,K,\xi_1,\ldots,\xi_K} = \bigl\{x \in \mathbb{R}^d \mid x \cdot \xi_i \le c_1^*(\xi_i) + \varepsilon,\ i = 1,\ldots,K\bigr\}, \tag{4.4}$$

for some $\varepsilon > 0$, $K \in \mathbb{N}$, $\xi_1,\ldots,\xi_K \in S^{d-1}$. By (4.1) and Proposition 4.1, the set $\Upsilon_1 = \Upsilon_*$ is bounded and nonempty. Take an arbitrary closed set $\mathscr{C} \subset \mathbb{R}^d \setminus \Upsilon_*$, and consider the open set $\mathscr{O} := \mathbb{R}^d \setminus \mathscr{C} \supset \Upsilon_* = \Upsilon_1$. Then, by [32, Lemma 7.2], there exist $\varepsilon > 0$, $K \in \mathbb{N}$, $\xi_1,\ldots,\xi_K \in S^{d-1}$ and a set $M \in \mathcal{M}$ of the form (4.4), such that

$$\Upsilon_* = \Upsilon_1 \subset M \subset \mathscr{O}. \tag{4.5}$$

Choose now

$$\mathscr{O}_{\xi_i} = \Bigl\{x \in \mathbb{R}^d \Bigm| x \cdot \xi_i < c_1^*(\xi_i) + \frac{\varepsilon}{2}\Bigr\} \supset \Upsilon_{1,\xi_i}, \quad 1 \le i \le K.$$

Then, by (4.5),

$$\Upsilon_* = \Upsilon_1 = \bigcap_{\xi \in S^{d-1}} \Upsilon_{1,\xi} \subset \bigcap_{i=1}^K \Upsilon_{1,\xi_i} \subset \bigcap_{i=1}^K \mathscr{O}_{\xi_i} \subset M \subset \mathscr{O},$$

and, therefore,

$$\mathbb{R}^d \setminus \mathscr{O} \subset \bigcup_{i=1}^K (\mathbb{R}^d \setminus \mathscr{O}_{\xi_i}). \tag{4.6}$$

Denote

$$\nu_i := \lambda_*(\xi_i)\,\mathrm{dist}\,(\Upsilon_{1,\xi_i}, \mathbb{R}^d \setminus \mathscr{O}_{\xi_i}) = \lambda_*(\xi_i)\frac{\varepsilon}{2}, \quad 1 \le i \le K.$$

Then, by the second item of Theorem 1.3 and (4.6), one gets, for any $t > 0$,

$$\operatorname*{esssup}_{x \in t\mathscr{C}} u(x,t) = \operatorname*{esssup}_{x \notin t\mathscr{O}} u(x,t) \le \max_{1 \le i \le K} \operatorname*{esssup}_{x \notin t\mathscr{O}_{\xi_i}} u(x,t) \le De^{-\nu t},$$

with $\nu := \min\{\nu_i \mid 1 \le i \le K\}$, $D := \max\{\|u_0\|_{\lambda_*(\xi_i),\xi_i} \mid 1 \le i \le K\}$. $\square$

4.2 Convergence to θ

We proof, at first, the second item of Theorem 1.1 for uniformly continuous functions. Namely, we assume that $u_0 \in C_\theta \cap C_{ub}(\mathbb{R}^d)$, $u_0 \not\equiv 0$, cf. (2.4), and we will prove, under assumptions (A1), (A2), (A4)–(A6) that, for any compact set $\mathscr{C} \subset \operatorname{int}(\Upsilon_*) = \operatorname{int}(\Upsilon_1)$,

$$\lim_{t\to\infty} \min_{x \in t\mathscr{C}} u(x,t) = \theta. \tag{4.7}$$

To do this, in Proposition 4.12, we apply results of [32] for discrete time, to prove (4.7) for continuous time, provided that u_0 is separated from 0 on a large enough set. Then we will use the hair-trigger effect (Theorem 2.5), which implies that $u(x,\tau)$ is separated from 0 on an arbitrary large set (shifted by $\tau\mathfrak{m}$) for big enough $\tau > 0$. Combining these results, we will get (4.7) for an arbitrary $u_0 \in C_\theta \cap C_{ub}(\mathbb{R}^d)$, $u_0 \not\equiv 0$. Finally, by the comparison principle, we will get the second item of Theorem 1.1 for $u_0 \in E_\theta^+$.

We start with the following Weinberger's result (rephrased in our settings). Note that, under (A1)–(A6), $\Upsilon_T \neq \emptyset$, $T > 0$. Indeed, if there exists $\xi \in S^{d-1}$, such that (A3$_\xi$) holds, then the result above follows from Proposition 4.1 and (2.13). Otherwise, $\Upsilon_* = \mathbb{R}^d$ and (2.13) yields the statement.

Lemma 4.8 (cf. [32, Theorem 6.2]). *Let* (A1)*,* (A2)*,* (A4)–(A6) *hold. Let $u_0 \in C_\theta$ and $T > 0$ be arbitrary, and Q_T be given by* (2.1) *(in particular, Q_T satisfies the properties* (Q1)–(Q5) *of Theorem 2.1). Define*

$$u_{n+1}(x) := (Q_T u_n)(x), \quad n \geq 0. \tag{4.8}$$

Then, for any compact set $\mathscr{C}_T \subset \operatorname{int}(\Upsilon_T)$ and for any $\sigma \in (0,\theta)$, one can choose a radius $r_\sigma = r_\sigma(Q_T, \mathscr{C}_T)$, such that

$$u_0(x) \geq \sigma, \quad x \in B_{r_\sigma}(0), \tag{4.9}$$

implies

$$\lim_{n\to\infty} \min_{x\in n\mathscr{C}_T} u_n(x) = \theta. \tag{4.10}$$

Remark 4.9. By the proof of [32, Theorem 6.2], the radius $r_\sigma(Q_T, \mathscr{C}_T)$ is not defined uniquely. In the sequel, $r_\sigma(Q_T, \mathscr{C}_T)$ means just a radius which fulfills the assertion of Lemma 4.8 for the chosen Q_T and $\mathscr{C}_T$, rather than a function of Q_T and $\mathscr{C}_T$.

Remark 4.10. It is worth noting, that, by (2.1) and the uniqueness of the solution to (1.1), the iteration (4.8) is just given by

$$u_n(x) = u(x, nT), \quad x \in \mathbb{R}^d, n \in \mathbb{N} \cup \{0\}. \tag{4.11}$$

Therefore, (4.10) with $T = 1$ yields (4.7), for $\mathbb{N} \ni t \to \infty$, namely,

$$\lim_{n\to\infty} \min_{x\in n\mathscr{C}} u(x,n) = \theta, \tag{4.12}$$

provided that (4.9) holds with $r_\sigma = r_\sigma(Q_1, \mathscr{C})$, $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$.

Lemma 4.11. *Let* (A1)*,* (A2)*,* (A4)*–*(A6) *hold. Fix a $\sigma \in (0,\theta)$ and a compact set $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$. Let $u_0 \in C_\theta$ be such that $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1,\mathscr{C})}(0)$. Then, for any $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \min_{x\in \frac{n}{k}\mathscr{C}} u\Big(x, \frac{n}{k}\Big) = \theta. \tag{4.13}$$

Proof. Since $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$, one can choose a compact set $\tilde{\mathscr{C}} \subset \mathrm{int}(\Upsilon_1)$ such that

$$\mathscr{C} \subset \mathrm{int}(\tilde{\mathscr{C}}). \tag{4.14}$$

By (4.11) and Lemma 4.8 (with $T = 1$), the assumption $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1,\mathscr{C})}(0)$ implies (4.12). Fix $k \in \mathbb{N}$, take $p = \frac{1}{k}$; then choose and fix the radius $r_\sigma\big(Q_p, p\tilde{\mathscr{C}}\big)$. By (4.12), there exists an $N = N(k) \in \mathbb{N}$, such that

$$\begin{gathered} u(x,N) \geq \sigma, \quad x \in N\mathscr{C}, \\ B_{r_\sigma(Q_p,p\tilde{\mathscr{C}})}(0) \subset N\mathscr{C}. \end{gathered}$$

Apply now Lemma 4.8, with $u_0(x) = u(x,N)$, $x \in \mathbb{R}^d$, $T = p$, and

$$\mathscr{C}_T = \mathscr{C}_p := p\tilde{\mathscr{C}} \subset p\,\mathrm{int}(\Upsilon_1) = \mathrm{int}(\Upsilon_p),$$

as, by (4.1), $p\Upsilon_1 = \Upsilon_p$. We will get then

$$\lim_{n\to\infty} \min_{x\in np\tilde{\mathscr{C}}} u(x, N+np) = \theta. \tag{4.15}$$

By (4.14), there exists $M \in \mathbb{N}$ such that one has

$$\Big(\frac{N}{n} + p\Big)\mathscr{C} \subset p\tilde{\mathscr{C}}, \quad n \geq M. \tag{4.16}$$

Therefore, by (4.16), one gets, for $n \geq M$,

$$\begin{aligned} \min_{x\in np\tilde{\mathscr{C}}} u(x,N+np) &\leq \min_{x\in n(\frac{N}{n}+p)\mathscr{C}} u(x,N+np) \\ &= \min_{x\in (Nk+n)\frac{1}{k}\mathscr{C}} u\Big(x, (Nk+n)\frac{1}{k}\Big) \leq \theta. \end{aligned} \tag{4.17}$$

By (4.15) and (4.17), one gets the statement. $\square$

Now, one can prove (4.7), under an assumption on the initial condition.

Proposition 4.12. *Let* (A1)*,* (A2)*,* (A4)*–*(A6) *hold. Fix a $\sigma \in (0,\theta)$ and a compact set $\mathscr{C} \subset \mathrm{int}(\Upsilon_1)$. Let $u_0 \in C_\theta \cap C_{ub}(\mathbb{R}^d)$ be such that $u_0(x) \geq \sigma$, $x \in B_{r_\sigma(Q_1,\mathscr{C})}(0)$, and $u \in \mathcal{X}_\infty$ be the corresponding solution to* (1.1)*. Then* (4.7) *holds.*

Proof. Suppose (4.7) were false. Then, there exist $\varepsilon > 0$ and a sequence $t_N \to \infty$, such that $\min\limits_{x\in t_N\mathscr{C}} u(x,t_N) < \theta - \varepsilon$, $n \in \mathbb{N}$. Since $t_N\mathscr{C}$ is a compact set and, by (Q1) in Theorem 2.1,

$$u(\cdot,t) \in C_\theta \cap C_{ub}(\mathbb{R}^d), \quad t \geq 0, \tag{4.18}$$

there exists $x_N \in t_N\mathscr{C}$, such that

$$u(x_N, t_N) < \theta - \varepsilon, \quad n \in \mathbb{N}. \tag{4.19}$$

Next, by (4.18) and [15, Proposition 5.1], there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $x', x'' \in \mathbb{R}^d$ and for all $t', t'' > 0$, with $|x' - x''| + |t' - t''| < \delta$, one has

$$|u(x', t') - u(x'', t'')| < \frac{\varepsilon}{2}. \tag{4.20}$$

Since $\mathscr{C}$ is a compact, $p(\mathscr{C}) := \sup\limits_{x\in\mathscr{C}} \|x\| < \infty$. Choose $k \in \mathbb{N}$, such that $\frac{1}{k} < \frac{\delta}{1+p(\mathscr{C})}$. By (4.13), there exists $M(k) \in \mathbb{N}$, such that, for all $n \geq M(k)$,

$$u\Bigl(x, \frac{n}{k}\Bigr) > \theta - \frac{\varepsilon}{2}, \quad x \in \frac{n}{k}\mathscr{C}. \tag{4.21}$$

Choose $N > N_0$ big enough to ensure $t_N > \frac{M(k)}{k}$. Then, there exists $n \geq M(k)$, such that $t_N \in \bigl[\frac{n}{k}, \frac{n+1}{k}\bigr)$. Hence

$$\Bigl|t_N - \frac{n}{k}\Bigr| < \frac{1}{k} < \frac{\delta}{1 + p(\mathscr{C})}. \tag{4.22}$$

Next, for the chosen N, there exists $y_N \in \mathscr{C}$, such that $x_N = t_N y_N$. Set $t' = t_N$, $t'' = \frac{n}{k}$, $x' = x_N = t_N y_N$, and $x'' = \frac{n}{k} y_N$. Then, by (4.22),

$$|t' - t''| + |x' - x''| = \Bigl|t_N - \frac{n}{k}\Bigr|\bigl(1 + |y_N|\bigr) < \delta.$$

Therefore, one can apply (4.20). Combining this with (4.19), one gets

$$u\Bigl(\frac{n}{k}y_N, \frac{n}{k}\Bigr) = u\Bigl(\frac{n}{k}y_N, \frac{n}{k}\Bigr) - u(t_N y_N, t_N) + u(x_N, t_N) < \frac{\varepsilon}{2} + \theta - \varepsilon = \theta - \frac{\varepsilon}{2},$$

that contradicts (4.21), as $\frac{n}{k} y_N \in \frac{n}{k}\mathscr{C}$. Hence the statement is proved. □

Now, we are ready to prove the second item of Theorem 1.1.

Proposition 4.13. *Let the conditions* (A1), (A2), (A4)–(A6) *hold. Let* $u_0 \in E_\theta^+$ *be such that there exist* $x_0 \in \mathbb{R}^d$, $\eta > 0$, $r > 0$, *with* $u_0(x) \geq \eta$ *for a.a.* $x \in B_r(x_0)$*; and let* $u \in \mathcal{X}_\infty$ *be the corresponding classical solution to* (1.1) *on* $\mathbb{R}_+$. *Then, for any compact set* $\mathscr{C} \subset \operatorname{int}(\Upsilon_*)$, *the convergence* (1.8) *holds.*

Proof. At first, we suppose that $u_0 \in C_\theta \cap C_{ub}(\mathbb{R}^d)$. For $u_0 \equiv \theta$, the statement is trivial. Hence let $u_0 \not\equiv \theta$, $u_0 \not\equiv 0$. Recall that, (A6) implies (A8).

Let $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$ be an arbitrary compact set. It is well-known, that the distance between disjoint compact and closed sets is positive; in particular, one can consider the compact $\mathscr{C}$ and the closure of $\mathbb{R}^d \setminus \Upsilon_1$. Therefore, there exists a compact set $\mathscr{K} \subset \operatorname{int}(\Upsilon_1)$, such that $\mathscr{C} \subset \operatorname{int}(\mathscr{K})$. Let $\delta_0 > 0$ be the distance between $\mathscr{C}$ and the closure of $\mathbb{R}^d \setminus \mathscr{K}$.

Choose any $\sigma \in (0, \theta)$ and consider a radius $r_\sigma = r_\sigma(Q_1, \mathscr{K})$ which fulfills Proposition 4.12, cf. Remark 4.9. By Theorem 2.5, there exists $t_1 > 0$, such that

$$u(x + t_1\mathfrak{m}, t_1) \geq \sigma, \quad |x| \leq r_\sigma.$$

We apply now Proposition 4.12 (with $\mathscr{C}$ replaced by $\mathscr{K}$) to the equation (1.1) with

$$u_0(x) := u(x + t_1\mathfrak{m}, t_1), \quad x \in \mathbb{R}^d$$

By (4.7) and the uniqueness arguments, we will have then

$$\lim_{t\to\infty} \min_{x\in t\mathscr{K}} u(x + t_1\mathfrak{m}, t + t_1) = \theta. \tag{4.23}$$

By (4.23), for any $\varepsilon > 0$, there exists $t_2 > 0$ such that, for all $t > t_1 + t_2 =: t_3 > 0$ and for all $y \in \mathscr{K}$,

$$u\bigl((t - t_1)y + t_1\mathfrak{m}, t\bigr) > \theta - \varepsilon \tag{4.24}$$

Without loss of generality we can assume that t_2 is big enough to ensure

$$t_1 \max_{x\in\mathscr{C}} |x| + t_1|\mathfrak{m}| < \delta_0 t_2. \tag{4.25}$$

Then, for any $x \in \mathscr{C}$ and for any $t > t_3$, the vector

$$y(x, t) := \frac{tx - t_1\mathfrak{m}}{t - t_1}$$

is such that

$$|y(x, t) - x| = \frac{\bigl|t_1x - t_1\mathfrak{m}\bigr|}{t - t_1} < \delta_0,$$

where we used (4.25). Therefore, $y(x, t) \in \mathscr{K}$, for all $x \in \mathscr{C}$ and $t > t_3$, and hence (4.24), being applied for any such $y(x, t)$, yields

$$u(tx, t) > \theta - \varepsilon, \quad x \in \mathscr{C},\ t > t_3,$$

that fulfills the proof of (4.7) for $u_0 \in C_\theta \cap C_{ub}(\mathbb{R}^d)$.

Let now $u_0 \in E_\theta^+$ satisfy the assumptions. Then there exists a function $v_0 \in C_\theta \cap C_{ub}(\mathbb{R}^d) \subset E_\theta^+$, $v_0 \not\equiv 0$, such that $u_0(x) \geq v_0(x)$, for a.a. $x \in \mathbb{R}^d$. Next, by Proposition 2.2, $u(x, t) \geq v(x, t)$, for a.a. $x \in \mathbb{R}^d$, and for all $t \geq 0$, where $v \in \mathcal{X}_\infty$ is the corresponding to v_0 solution to (1.1). Then, by the proved above, we will get (4.7) for v, with the same Υ_1, cf. (Q1) of Theorem 2.1. As a result, the evident inequality

$$\min_{x\in t\mathscr{C}} v(x, t) \leq \operatorname*{essinf}_{x\in t\mathscr{C}} u(x, t) \leq \theta$$

implies (1.8). The statement is fully proved now. □

Now one can prove Proposition 1.7.

Proof of Proposition 1.7. Suppose that, in contrast, for some $\xi \in S^{d-1}$, $c \in \mathbb{R}$, and $\psi \in \mathcal{M}_\theta(\mathbb{R})$, (1.16) holds. Then $u_0(x) = \psi(x\cdot\xi)$ satisfies the assumptions of the first statement. Take a compact set $\mathscr{K} \subset \mathbb{R}^d$, such that $c_1 := \max_{y\in\mathscr{K}} y \cdot \xi > c$. Then (1.8) implies

$$\begin{aligned}\theta &= \lim_{t\to\infty} \operatorname*{essinf}_{x\in t\mathscr{K}} \psi(x \cdot \xi - ct) = \lim_{t\to\infty} \operatorname*{essinf}_{y\in\mathscr{K}} \psi\bigl(t(y \cdot \xi - c)\bigr)\\ &= \lim_{t\to\infty} \psi\bigl(t(c_1 - c)\bigr) = 0,\end{aligned}$$

where we used that ψ is decreasing. One gets a contradiction which proves the second statement. □

Another important application of the second item in Theorem 1.1 is that there are not stationary solutions $u \geq 0$ to (1.1) (i.e. solutions with $\frac{\partial}{\partial t}u = 0$), except $u \equiv 0$ and $u \equiv \theta$, provided that the origin belongs to $\mathrm{int}(\Upsilon_*)$.

Proposition 4.14. *Let* (A1)–(A6) *hold. If* $\varkappa_\ell = 0$ *in* (1.1), *we assume, additionally, that there exists* $r_0 > 0$ *such that*

$$\alpha := \inf_{|x|\leq r_0} a^-(x) > 0. \tag{4.26}$$

Let also the origin belong to $\mathrm{int}(\Upsilon_*)$. *Then there exist only two non-negative stationary solutions to* (1.1) *in* E, *namely,* $u \equiv 0$ *and* $u \equiv \theta$.

Proof. Since $\frac{\partial}{\partial t}u = 0$, one gets from (1.1) that

$$u(x) = \frac{\pm\sqrt{D(x)} - \bigl(m + B(x)\bigr)}{\varkappa_\ell}, \quad x \in \mathbb{R}^d, \tag{4.27}$$

where

$$\begin{aligned} A(x) &= \varkappa^+(a^+ * u)(x),\ B(x) = \varkappa_{n\ell}(a^- * u)(x), \\ D(x) &= \bigl(m + B(x)\bigr)^2 + 4\varkappa_\ell A(x) \geq m > 0. \end{aligned}$$

Then, by [12, Lemma 2.1], one easily gets that $u \in C_{ub}(\mathbb{R}^d)$.

Denote $M := \|u\| = \sup\limits_{x\in\mathbb{R}^d} u(x)$. We are going to prove now that $M \leq \theta$. On the contrary, suppose that $M > \theta$. One can rewrite (4.27) as follows:

$$\begin{aligned} mu(x) + \varkappa_\ell u^2(x) + \varkappa_{n\ell}(a^- * u)(x)(u(x) - \theta) \\ = (J_\theta * u)(x) \leq M(\varkappa^+ - \varkappa_{n\ell}\theta), \end{aligned} \tag{4.28}$$

where

$$J_\theta(x) := \varkappa^+ a^+(x) - \theta\varkappa_{n\ell}a^-(x) \geq 0,$$

and hence $\int_{\mathbb{R}^d} J_\theta(x)\,dx = \varkappa^+ - \varkappa_{n\ell}\theta$.

Choose a sequence $x_n \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that $u(x_n) \to M$, $n \to \infty$. Substitute x_n to the inequality (4.28) and pass $n \to \infty$. Since $M > \theta$ and $u \geq 0$, one gets then that $(a^- * u)(x_n) \to 0$, $n \to \infty$. Passing to a subsequence of $\{x_n\}$ and keeping the same notation, for simplicity, one gets that

$$(a^- * u)(x_n) \leq \frac{1}{n},\ n \geq 1.$$

For all $n \geq r_0^{-2d}$, set $r_n := n^{-\frac{1}{2d}} \leq r_0$; then the inequality (4.26) holds, for any $x \in B_{r_n}(0)$, and hence

$$\frac{1}{n} \geq (a^- * u)(x_n) \geq \alpha(1\!\!1_{B_{r_n}(0)} * u)(x_n) \geq \alpha V_d(r_n) \min_{x\in B_{r_n}(x_n)} u(x), \tag{4.29}$$

where $V_d(R)$ is a volume of a sphere with the radius $R > 0$ in $\mathbb{R}^d$. Since $V(r_n) = r_n^d V_d(1) = n^{-\frac{1}{2}}V_d(1)$, we have from (4.29), that, for any $n \geq r_0^{-2d}$, there exists $y_n \in B_{r_n}(x_n)$, such that

$$u(y_n) \leq \frac{1}{\alpha\sqrt{n}V_d(1)}.$$

Thus $u(y_n) \to 0$, $n \to \infty$. Recall that $u(x_n) \to M > 0$, $n \to \infty$, however, $|x_n - y_n| \leq r_n = n^{-\frac{1}{2d}}$, that may be arbitrary small. This contradicts the fact that $u \in C_{ub}(\mathbb{R}^d)$.

As a result, $0 \leq u(x) \leq \theta = M$, $x \in \mathbb{R}^d$. Let $u \not\equiv 0$. By the third item in Theorem 1.1, for any compact set $\mathscr{C} \subset \operatorname{int}(\Upsilon_1)$, $\min\limits_{x \in t\mathscr{C}} u(x) \to \theta$, $t \to \infty$, as $u(x,t) = u(x)$ now. Since $0 \in \operatorname{int}(\Upsilon_1)$, the latter convergence is obviously possible for $u \equiv \theta$ only. $\square$

Remark 4.15. It is worth noting that, by (2.11), (2.13), and (2.12), the assumption $0 \in \operatorname{int}(\Upsilon_1)$ implies that $c_*(\xi) \geq 0$, for all $\xi \in S^{d-1}$. It means that all traveling waves in all directions have nonnegative speeds only.

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